PUTNAM PRACTICE SET 6

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Problem 1. Let a and s be real numbers satisfying the following properties:

- $0 < a \leq 1$; and
- $s > 0$, but $s \neq 1$.

Prove that $\frac{1-s^a}{1-s} \leq (1+s)^{a-1}$.

Problem 2. Let $n > 3$ be an integer.

(1) Find all permutations σ of the set $\{1, 2, \ldots, n\}$ for which the sum
$$\sigma(1) \cdot \sigma(2) + \sigma(2) \cdot \sigma(3) + \cdots + \sigma(n-1) \cdot \sigma(n) + \sigma(n) \cdot \sigma(1)$$
is maximal.

(2) Find all permutations σ of the set $\{1, 2, \ldots, n\}$ for which the sum
$$\sigma(1) \cdot \sigma(2) + \sigma(2) \cdot \sigma(3) + \cdots + \sigma(n-1) \cdot \sigma(n) + \sigma(n) \cdot \sigma(1)$$
is minimal.

Problem 3. We consider a set S of finitely many disks in the cartesian plane (of arbitrary centers and arbitrary radii) and we let A be the area of the region represented by their union. Prove that there exists a subset $S_0 \subseteq S$ satisfying the following two properties:

- any two disks from S_0 are disjoint.
- the sum of the areas of the disks from S_0 is at least $\frac{A}{9}$.

Problem 4. Let $\{u_n\}_{n \geq 1}$ be a recurrence sequence defined by $u_{n+1} = \frac{\sqrt[3]{64u_n + 15}}{4}$ for each $n \geq 1$. Find $\lim_{n \to \infty} u_n$.